WANDERING INTERVALS AND ABSOLUTELY CONTINUOUS INVARIANT PROBABILITY MEASURES OF INTERVAL MAPS

HONGFEI CUI [a,b] AND YIMING DING [a]

[a] Wuhan Institute of Physics and Mathematics
The Chinese Academy of Sciences, Wuhan 430071, China

[b] Graduate School of the Chinese Academy of Sciences
The Chinese Academy of Sciences, Beijing 10049, China
Email Address: cuihongfei05@mails.gucas.ac.cn, ding@wipm.ac.cn

ABSTRACT. For piecewise C^1 interval maps possibly containing critical points and discontinuities with negative Schwarzian derivative, under two summability conditions on the growth of the derivative and recurrence along critical orbits, we prove

(1) the nonexistence of wandering intervals,
(2) the existence of absolutely continuous invariant measures, and
(3) the bounded backward contraction property.

The proofs are based on the method of proving the existence of absolutely continuous invariant measures of unimodal map, developed by Nowicki and van Strien.

1. INTRODUCTION AND MAIN RESULTS

The concept of wandering intervals plays an important role in studying dynamical behavior of non-uniformly hyperbolic dynamical system. In the area of interval dynamics, most important results are related to the absence of wandering intervals. Our main aim in this paper is to obtain a condition on the orbits of the critical values to ensure the absence of wandering intervals for piecewise C^1 interval maps with critical points and discontinuities, and to give a sufficient condition to the nonexistence of wandering intervals for multimodal maps whose orders of critical points are different to the left and to the right.

The motivation to show nonexistence of wandering intervals are well known. Firstly, it is relevant to the isomorphism problem of dynamical system. In the 1880s, Poincaré proved that each orientation preserving homeomorphism of the circle without periodic points is semi-conjugate to an irrational rotation. Denjoy showed that for the C^2 diffeomorphism of the circle without periodic points such wandering interval cannot exist, and this semiconjugacy is indeed a conjugacy. Analogue of Denjoy's theory also holds for a C^3 unimodal map f with a non-flat critical point (whose orders are equal to the left and to the right) and negative Schwarzian derivative,

Mathematical classification (2000): 37E05, 28D05, 37D25.

Keywords and Phrases: wandering interval, contracting Lorenz map, absolutely continuous invariant probability measure, bounded backward contraction property.

This work was partly supported from the NSFC grant 60534080.

it was shown that f admits no wandering intervals, and it is conjugate to a quadratic map f_μ ($f_\mu = \mu x(1-x), \mu \in (0,4)$) with some value of the parameter μ [11] if it has no periodic attractor. Secondly, the nonexistence of wandering intervals is also relevant to the dynamics of single dynamical system. A very remarkable result about iterations of rational maps of the Riemann sphere proved by Sullivan is that there are no wandering components in Fatou set, i.e., all of the connected components of Fatou set of a rational map are eventually periodic [29]. The Julia -Fatou-Sullivan theory for the dynamics of rational map is also valid for C^2 multimodal maps with non-flat critical points whose orders are equal to the left and to the right, it was shown that there are no wandering intervals for such maps [22].

Besides the above results, the nonexistence of wandering intervals was proved by a series remarkable works: by Schwartz for continuous piecewise C^1 interval maps f with $\log|Df|$ satisfying Lipschitz condition in [27], by Guckenheimer for maps which everywhere have negative Schwarzian derivative in [15], by Yoccoz for C^∞ homeomorphisms of the circle with non-flat critical points in [31], by Blokh and Lyubich for smooth interval maps with all critical points are turning points in [3] and [20], and by van Strien and Vargas for general multimodal maps whose orders of each critical point are equal from both sides in [30].

However, the wandering intervals may exist for some maps. For the continuous case, Denjoy's counterexample tell us that a C^1 differeomorphism on the circle may have a wandering interval [11], and there are C^∞ maps on compact interval with flat critical points which have wandering intervals [17]. For the discontinuous case, there are wandering intervals for a Lorenz map if it can be renormalized to be a gap map with irrational rotation number, Berry and Mestel showed that wandering intervals exist only in this case for the Lorenz map f with $\log|Df|$ satisfying the Lipschitz condition in [2]. There exists affine interval exchange transformations which have wandering intervals [9] [16].

From the above examples, we cannot expect that there are no wandering intervals for discontinuous interval maps with critical points like continuous case under some general conditions. On the other hand, the known results of nonexistence of wandering intervals for continuous multimodal maps always require the orders of each critical point are the same to the left and to the right. Blokh had asked whether wandering intervals can exist for unimodal maps which are smooth except at their critical point and the critical order is different to the left and to the right (this question is quoted in [12], or [4] for a general case). In this paper, we consider the nonexistence of wandering intervals of interval maps with some additional conditions, and give a sufficient condition to answer Blokh's question for general maps in some sense.

We now give the precise statement of our main result. An interval J is a *wandering interval* for a map $f : M \to M$ if it satisfies:

(a) its forward iterates $J, f(J), ..., f^n(J)$ are all disjoint for all $n > 0$;

(b) J is not contained in the basin of attraction of an attracting periodic orbit;

(c) $f^n|_J$ is a homeomorphism for all $n > 0$.

Let $\mathcal{A}$ denote the class of interval maps satisfying conditions 1 and 2 listed below. Then we have the following

Theorem A. *Wandering intervals can not exist for each map in $\mathcal{A}$.*

1. Let M be a compact interval $[0,1]$, $f : M \longrightarrow M$ be a piecewise C^1 interval map with negative Schwarzian derivative. This means that there exists a finite set C such that f is a diffeomorphism on each component of $M \setminus C$, and admits a continuous extension to the boundary so that the left and the right limits $f(c\pm) = \lim_{x\to c\pm} f(x)$ exist. We always regard each $c \in C$ as two points: $c+$ and $c-$, the concrete values depend on the corresponding one-side neighborhoods we considered. We assume that each $c \in C$ has two one-side critical orders $l(c\pm) \in [1,\infty)$, this means that

$$|Df(x)| \approx |x-c|^{l(c\pm)-1}, \quad |f(x) - f(c\pm)| \approx |x-c|^{l(c\pm)},$$

for x in the corresponding one-side neighborhood of c, where we say $f \approx g$ if the ratio f/g is bounded above and below uniformly in its domain. When we use $l(c)$, it may be either $l(c+)$ or $l(c-)$, and the concrete value can be easily understood from the context. If $l(c) > 1$, we say that c is a critical point, if $l(c) = 1$, we say that c is a bounded derivative point. Note that c may be a critical point on one side and is a bounded derivative point. When there is no possibility of confusion, each point $c \in C$ will be called a critical point without distinguishing whether c is really a critical point with $l(c) > 1$, or c is a bounded derivative point with $l(c) = 1$.

We also assume that f is with negative Schwarzian derivative outside of C, i.e., $|f'|^{-\frac{1}{2}}$ is a convex function on each component of $M \setminus C$.

2. We suppose that f satisfies the following summability conditions along the critical orbits. The first summability condition is

$$\sum_{n=1}^{\infty} \Big(\frac{|f^n(c) - \tilde{c}|^{l(\tilde{c})}}{|f^n(c) - \tilde{c}|^{l(c)} |Df^n(f(c))|} \Big)^{1/l(c)} < \infty, \quad \forall c \in C, \tag{1. 1}$$

where $\tilde{c}$ is the critical point closest to $f^n(c)$, and $l(c)$ and $l(\tilde{c})$ depend on the corresponding one-side neighborhoods. The second summability condition is

$$\sum_{n=1}^{\infty} \frac{1}{|Df^n(f(c))|^{1/l(c)}} < \infty, \quad \forall c \in C. \tag{1. 2}$$

Remark 1. According to the first summability condition, a map $f \in \mathcal{A}$ can not map its critical point to another critical point, i.e., $C \cap \cup_{n\geq 1} f^n(C) = \emptyset$, and it is easy to see that if all of the critical orders are equal, the first summability condition is equivalent to the second summability condition and $C \cap \cup_{n\geq 1} f^n(C) = \emptyset$. The above similar summability conditions along the critical orbits have been applied to show the existence of absolutely continuous invariant measure for unimodal maps in [25], for multimodal maps in [7] and for interval maps possible with critical points and singularities in [1] recently. The second summability condition is similar to the Nowicki-van Strien condition in [25].

Remark 2. Simple examples satisfying the conditions 1 and 2 listed above are the contracting Lorenz maps considered in [23] and [26], which were motivated by the study of the return map of the Lorenz equations near classical parameter values, see Figure 1. For Lorenz map f with $\log|Df|$ satisfying the Lipschitz condition, Berry and Mestel showed in [2] that wandering intervals exist if and only if it can be renormalized to be a gap map with irrational rotation number. This, together with Theorem A, implies that if a C^1 Lorenz map f with $l(c) = 1$ satisfying the summability condition $\sum_{n=1}^{\infty} \frac{1}{|Df^n(f(c))|} < \infty$ and the negative Schwarzian derivative condition, then it can not be renormalized to be a gap map with irrational rotation number. On the other hand, for C^2 Lorenz map f whose critical orders are greater than 1, it is conjectured in an old version of [21] (arxiv: math/9610222v1) and in [28] that f admits wandering intervals if and only if it can be renormalized to be a gap map with irrational rotation number, while Theorem A indicates that wandering intervals can not exist for Lorenz map f satisfying $\sum_{n=1}^{\infty} \frac{1}{|Df^n(f(c))|^{1/l(c)}} < \infty$.

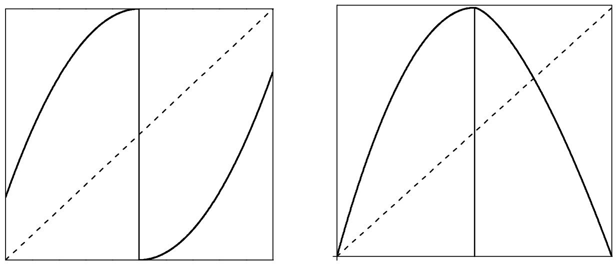

Figure 1: The contracting Lorenz map, and unimodal map with different critical orders.

In particular, for continuous maps in $\mathcal{A}$, we answer the Blokh's question in some sense partially.

Corollary 1. *Let f be a C^1 multimodal map with finite critical points whose orders may not equal to the left and to the right. If f satisfies the negative Schwarzian derivative condition and our summability conditions, then f admits no wandering intervals.*

Furthermore, we can prove the following Theorem B.

Theorem B. *Let f be a map in $\mathcal{A}$, then f admits an absolutely continuous invariant probability measure (acip). Moreover, if $l_{\max} > 1$, the density is in L^p for all $1 \le p < l_{\max}/(l_{\max} - 1)$, where $l_{\max}$ is the maximum of the orders of the critical points.*

We present some comments on Theorem B. At first, Theorem B improves a result in [23]. It was shown in [23] that for contracting Lorenz maps f satisfying the following conditions

(1) Outside of the unique critical point c $(l(c) > 1)$, f is of C^3 classes and with negative Schwarzian derivative, and with equal critical orders from both sides;

(2) (Increasing condition) $|Df^n(f(c\pm))| > \lambda^n$, for each $n \geq 1$ and some $\lambda > 1$;

(3) (Recurrence condition) $|f^{n-1}(f(c\pm)) - c| > \exp^{-\alpha n}$ for some α small enough, and for all $n \geq 1$,

then f admits an absolutely continuous invariant probability. While Theorem B tell us that it suffices to assume the summability condition

$$\sum_{n=1}^{\infty} \frac{1}{|Df^n(f(c))|^{1/l(c)}} < \infty$$

and the above condition (1) for the contracting Lorenz map. Moreover, Theorem B provides the regularity of the density of the acip.

Secondly, Theorem B is similar to a result in [1]. It was proved in [1] that for any piecewise C^2 interval map f possibly containing discontinuities and singularities $(0 < l(c) < 1)$ and satisfying an analytical condition (uniformly expanding away from the critical points) and

(1) *bounded backward contraction (BBC) property*, there is a constant $K > 0$ such that for all $\delta_0 > 0$, one can find $\delta \in (0, \delta_0)$ such that for a neighborhood N_δ (a concrete definition of N_δ is stated in section 4) of the critical points and each x,

$$|D^n f(x)| \geq K, \quad for \;\; n = \min\{i \geq 0; f^i(x) \in N_\delta\},$$

(2) *summability condition along the critical orbit*

$$(1.\ 3) \quad \sum_{n=1}^{\infty} \frac{-n \log |f^n(c) - C|}{|f^n(c) - C||Df^{n-1}(f(c))|^{1/(2l(c)-1)}} < \infty, \ \forall\, c \in C \ with \ l(c) > 1,$$

then f admits only finite number of absolutely continuous invariant (physical) probability measures. Note that the summability condition (1. 3) implies our first summability condition (1. 1) and the second summability condition (1. 2) in our argument for $l(c) > 1$, this is because for n large enough,

$$\Big(\frac{|f^n(c) - \tilde{c}|^{l(\tilde{c})}}{|f^n(c) - \tilde{c}|^{l(c)}|Df^n(f(c))|}\Big)^{1/l(c)} = \frac{|f^n(c) - \tilde{c}|^{l(\tilde{c})/l(c)-1}}{\big(|Df(f^n(c))| \; |Df^{n-1}(f(c))|\big)^{1/l(c)}}$$
$$\leq \frac{|f^n(c) - \tilde{c}|^{1/l(c)-1}}{|Df^{n-1}(f(c))|^{1/l(c)}} \leq \frac{-n \log |f^n(c) - C|}{|f^n(c) - C||Df^{n-1}(f(c))|^{1/(2l(c)-1)}},$$

by the definition of orders of critical points and $l(c) > 1$. As was stated in [1], in the special cases in which there are no singularities $(0 < l(c) < 1)$, the summability condition (1. 3) can not reduce to the summability condition assumed in [5], while our conditions can.

Moreover, we can show BBC property under our conditions.

Theorem C. *Let $f \in \mathcal{A}$, then f admits BBC property.*

As stated above, the BBC property is an assumption in [1], is also an assumption for the studying of the statistical properties such as decay of correlations and the Central Limit Theorem of interval maps in [13], and

an important Lemma in [5]. BBC property is true [24] for symmetric unimodal maps with negative Schwarzian derivative. It is also proved in [8] for multimodal case with the same critical orders of all critical points and

$$\lim_{n\to\infty} |Df^n(f(c))| = \infty, \quad \forall c \in C.$$

It is interesting that the motivation of the proof the BBC property in [8] is to show the existence of the acip, but in our argument we prove the BBC using the methods of the proof of the existence of the acip. We emphasize that the proof of the BBC property of mutlimodal case have to use the nonexistence of wandering intervals for multimodal map [Theorem 1.2 in [8]] and the same critical order of all of critical points. Using Theorem C, we can consider the decay of correlations and CLT for interval maps possibly containing critical points and discontinuities with negative Schwarzian derivative, under some summability conditions on the growth of the derivative and recurrence along critical orbits in our future work.

2. Ideas and organization of the proof

The proof of the nonexistence of wandering intervals of interval maps will be achieved by contradiction, so we suppose that there is a wandering interval J. According to Denjoy's original ideas, most of the proofs (see, for example [11], [22]) of the nonexistence of wandering intervals contain two ingredients: a topological one and an analytical one. The topological part is to the understanding of the detailed dynamics of interval map, while the analytical part is to estimate the distortion using the topological results. Then, either J would be attracted by a periodic orbit or one can get a conclusion that contradicts to the estimation of analytic aspect. Note that the condition that the orders of each critical point from both sides are equal is necessary in both ingredients of previous proofs.

Our proof is completely analytical, and has links to the method of the proof of the existence of the acip. The method, as in [25], is to estimate of the Lebesgue measure of $f^{-n}(A)$, where A is a small neighborhood of the critical points. The proof of Theorems can be divided into three steps:

Step 1. Show that if $f \in \mathcal{A}$ then there exists a constant $K_1 > 0$ so that $|f^{-n}(B(c,\epsilon))| \leq K_1\epsilon$, for any $c \in C$, $n \geq 0$ and small $\epsilon \geq 0$, where $B(c,\epsilon) = (c-\epsilon, c+\epsilon)$. This property relates the measure of preimages of a small neighborhood of the critical points to the measure of neighborhood of the critical points.

Step 2. Show that if $f \in \mathcal{A}$ and satisfies the above property in step 1, then there exists a constant $K_1 > 0$ so that $|f^{-n}(A)| \leq K_1|A|^{1/l_{\max}}$, where A is any Borel set. A more precise version of the above two steps is stated in Section 3. The proof seems complicated, but it is almost the same as the proof of the existence of the acip of unimodal map in [25], multimodal map in [7].

Step 3. In Section 4, we present the proofs of Theorem A, Theorem B, and Theorem C. Theorem A is a direct consequence of the property in step 2. The proof of Theorem B is classical when f is continuous. In fact, if f is continuous, the existence of acip follows from Proposition 2 and the compactness of the space of probability measure on M under the

weak star topology. This kind of argument can not be applied to our case directly because f may have discontinuities. We obtain the existence of acip by checking the uniformly integrability of the action of Frobenius-Perron operator on the constant function $\mathbf{1}$. To show Theorem C, the result in step 1 indicates that if $f \in \mathcal{A}$ then there exists a constant $K > 0$ such that $|Df^n(x)| > K$ for $f^n(x) \in C$, because the critical point will not be mapped into another critical point. This property implies that the derivatives of the preimages of the critical points are bounded away zero. Next we can find enough Koebe spaces so that we can relate the derivatives of the preimages of the critical point to the derivatives of the preimages of any point in the neighborhood of the critical point by the one-side Koebe principle. We refer [12] for the proof of the Koebe principle.

It is possible to improve these results. The negative Schwarzian derivative condition may be omitted, however the strategy in [18] and [30] made use of the nonexistence wandering intervals, and the method in [22] involves detailed analysis of topological dynamics of intervals maps. Note that the negative Schwarzian derivative condition rule out the existence of the singularities ($0 < l(c) < 1$), once one can get rid of this negative Schwarzian derivative condition, then the results in this paper can generalized to the interval maps with singularities and critical points. Secondly, general methods of proving the existence of the acip of smooth maps may work efficiently without using the result of the nonexistence of wandering intervals, the quite general conditions are known to guarantee the existence of the acip for smooth maps with a finite number of critical points recently in [6], but they used the results of the nonexistence of wandering intervals, too. However, it was conjectured by Araújo et al. in [1] that it is not possible to obtain a general result about the existence of acip in the presence of both critical points and singularities by assuming conditions on the derivatives growth of the critical points.

We denote by K_l the constant from the orders of the critical points, by K_o from the Koebe principle, and denote $|J|$ as the Lebesgue measure of J.

3. BACKWARD CONTRACTION

In this section, we will use the proof of the existence of acip of unimodal maps and multimodal maps, see [25] and [7], and we refer the chapter 5 of [12] for more details. We only give the main arguments, and do some modifications with the proof of multimodal maps in [7].

Proposition 1. *If $f \in \mathcal{A}$, then there exists a constant $K_1 > 0$ such that*

$$|f^{-n}(B(c,\epsilon))| \leq K_1\epsilon$$

for any $c \in C$, $n \geq 0$ and small $\epsilon \geq 0$, where $B(c,\epsilon) = (c-\epsilon, c+\epsilon)$.

Proof. Denote $E_n(c,\epsilon) := f^{-n}(B(c,\epsilon))$, we will divide the components I of $E_n(c,\epsilon)$ into three cases. Given $\sigma > 3\epsilon$, σ is a constant to be fixed by the summability conditions. Let $I \subset I' \subset I''$ be the components of $E_n(c,\epsilon), E_n(c,3\epsilon)$, and $E_n(c,\sigma)$ respectively. We distinguish three cases:

(1) $I \in \mathcal{R}_n$–the regular case, if f^n has no critical point in I''.
(2) $I \in \mathcal{S}_n$–the sliding case, if f^n has a critical point in I'' but not in I'.
(3) $I \in \mathcal{T}_n$–the transport case, if f^n contains a critical point in I'.

The regular case: Suppose $I \in \mathcal{R}_n(c)$, we know that $f^n(I'')$ contains a 1-scaled neighborhood of $f^n(I)$, by the Koebe principle there exists a constant $K_{\mathcal{R}} > 0$ such that

$$\frac{2\sigma}{|I''|} = \frac{|f^n(I'')|}{|I''|} \leq K_{\mathcal{R}} \frac{|f^n(I)|}{|I|} \leq K_{\mathcal{R}} \frac{2\epsilon}{|I|}.$$

We choose $K_{\mathcal{R}}$ big enough so that this holds for all critical point $c \in C$. Then we have

$$\sum_{I \in \mathcal{R}_n(c)} |I| \leq K_{\mathcal{R}} \frac{\epsilon}{\sigma} \sum_{I \in \mathcal{R}_n(c)} |I''| \leq K_{\mathcal{R}} \frac{\epsilon}{\sigma}. \tag{3. 4}$$

We shall show Proposition 1 by induction. The induction hypothesis is

$$|E_k(c, \epsilon)| \leq \frac{3K_{\mathcal{R}}}{\sigma} \epsilon \tag{3. 5}$$

for all $0 < \epsilon \leq \frac{\sigma}{3}$, $c \in C$, and $k < n$.

The above inequality (3. 5) is true for $n = 2$, because for σ sufficiently small, there are only regular cases.

In what follows we shall prove the inequality (3. 5) holds for $k = n$ for the sliding case and for the transport case.

Let $V(\sigma) = \min\{k \geq 1, |f^k(\tilde{c}) - C| < \sigma \quad for \quad some\ \tilde{c} \in C\}$, observe that $V(\sigma) \to \infty$ as $\sigma \to 0$ because f can not map a critical point to another critical point by the summability conditions.

The sliding case: Let $I \in \mathcal{S}_n(c)$, $T^0 \supset I$ be the maximal interval on which f^n is a diffeomorphism. Denote $T_0 = f^n(T^0) \supset f^n(I) = I_0 = B(c, \epsilon)$, R_0 and A_0 be the components of $T_0 \setminus A_0$ and $|R_0| \geq |A_0|$, denote $T^0 := [\alpha_0, \alpha_{-1}]$ with that $f^n(\alpha_0) \in \partial A_0$. Since $I \in \mathcal{S}_n(c)$, we have $|R_0| \geq |A_0| \geq |I_0|$. We shall construct a sequence $n = n_0 > n_1 > ... \geq 0$ and a nested sequence of intervals

$$T^s \supset ... \supset T^1 \supset T^0 = T \tag{3. 6}$$

as following:

(1) Choose $0 < n_1 < n$ such that $f^{n_1}(\alpha_0) \in C$, let $T^1 := [\alpha_1, \alpha_0]$ be the maximal interval containing T^0 on which f^{n_1} is a diffeomorphism. Assume that n_{i-1} and $T^{i-1} = [\alpha_{i-2}, \alpha_{i-1}]$ are defined, define $n_i < n_{i-1}$ such that $f^{n_i}(\alpha_{i-1}) \in C$ and let $T^i := [\alpha_i, \alpha_{i-1}]$ be the maximal interval which contains T^{i-1} and on which f^{n_i} is a diffeomorphism. Note that for $i \geq 1$, T^i and T^{i-1} have a precise common boundary α_i and that $I \subset T^0 \subset ... \subset T^i$.

(2) Let $k_i = n_i - n_{i+1}$, $I_{i+1} = f^{n_{i+1}}(I)$, $T_{i+1} = f^{n_{i+1}}(T^{i+1})$, R_{i+1} be the component of $T_{i+1} \setminus I_{i+1}$ and contains a critical point in its closure, and A_i be another component. Let $L_{i+1} \subset A_{i+1}$ be the interval adjacent to I_{i+1} and satisfying $f^{k_i}(L_{i+1}) = R_i$, we conclude the following relationships between the above intervals:

$$f^{k_i}(I_{i+1}) = I_i, f^{k_i}(R_{i+1}) = A_i, f^{k_i}(L_{i+1}) = R_i.$$

(3) The above construction stops at $n_s = 0$, when $|I_s| > |R_s|$ or when $|I_s| \leq |R_s| \leq |A_s|$.

Note that $|I_i| \leq |R_i|$ and $|A_i| < |R_i|$ for $0 \leq i \leq s - 1$.

The following Lemma is to control $|I_s|$.

Lemma 1. *(Proposition 1.3.1 in [7]) There exists $K_2 > 1$ such that*

$$|I_s| \le |f^n(I)| \prod_{i=0}^{s-1} K_2 \Big(\frac{|f^{k_i}(c_{i+1}) - \tilde{c}|^{l(\tilde{c})-l(c_{i+1})}}{|Df^{k_i}(f(c_{i+1}))|} \Big)^{1/l(c_{i+1})}, \tag{3. 7}$$

where $\tilde{c}$ is the critical point closest to $f^{k_i}(c_i)$, c_{i+1} is the critical point in ∂R_{i+1}.

Proof. This Lemma is almost the same as Proposition 1.3.1 in [7]. Since the proof of the Lemma only uses the Koebe principle, orders of the critical points and $|A_i| < |R_i|$ for $0 \le i \le s-1$, the result is also ture for $f \in \mathcal{A}$. □

Now we compare I with I_s. According to the stopping rules, we consider three cases:

(1) If $n_s = 0$, then $I = I_s$ by the definition of I_s, it follows that

$$I \subset E_0(c_s, \frac{1}{2}|I_s|). \tag{3. 8}$$

(2) If $|I_s| > |R_s|$, then we have

$$I \subset f^{-n_s}(I_s) \subset f^{-n_s}(I_s \cup R_s) \subset E_{n_s}(c_s, 2|I_s|). \tag{3. 9}$$

(3) If $|I_s| \le |R_s| \le |A_s|$, we use the 'sliding' technology developed by Nowicki and van Strien in [25]. Let $J \subset T^s$ be an interval such that $|J| = |I|$ and $G := f^{n_s}(J) \subset I_s \cup R_s$ be adjacent to c_s, since $|I_s| \le |R_s| \le |A_s|$, there exists constant K_o such that

$$\frac{|G|}{|J|} \le K_o \frac{|f^{n_s}(I)|}{|I|}$$

by the one-side Koebe principle. Since $|I| = |J|$, this gives that $|G| \le K_o |f^{n_s}(I)|$. Then

$$J \subset E_{n_s}(c_s, K_o|I_s|), \text{ and } |I| \le |E_{n_s}(c_s, K_o|I_s|)|. \tag{3. 10}$$

Lemma 2. *One can choose σ so small that the interval T_i in (3. 6) have size less than σ for $0 \le i < s$.*

Proof. Because $v(\sigma) \to \infty$ as $\sigma \to 0$ and $\Big(\frac{|f^n(c)-\tilde{c}|^{l(\tilde{c})}}{|f^n(c)-\tilde{c}|^{l(c)}|Df^n(f(c))|} \Big)^{1/l(c)} \to 0$ as $n \to \infty$, we can choose σ small enough so that for each $n \ge v(\sigma)$ one has

$$\Big(\frac{|f^n(c) - \tilde{c}|^{l(\tilde{c})}}{|f^n(c) - \tilde{c}|^{l(c)} |Df^n(f(c))|} \Big)^{1/l(c)} \le \frac{1}{K},$$

where K is a positive constant depending on $K_l, K_o, l(c)$, and the finiteness of the number of critical points. Let us show by induction that for small σ, $|R_i \cup I_i \cup A_i| \le \sigma$ for $i = 0, ..., s-1$. Assume that $s \ge 2$, $|T_0| \le \sigma$ follows from the definition. Observe that

$$|A_0| \le |f^{k_0}(c_1) - c_0| \le |A_0 \cup I_0| \le 2|A_0|. \tag{3. 11}$$

We have

$$
\begin{aligned}
|T_1| &= |R_1 \cup I_1 \cup A_1| \le 3|R_1| \\
&\le 3{K_l}^{1/l(c_1)}|f(R_1)|^{1/l(c_1)} \quad (oder\ of\ critical\ point) \\
&\le 3{K_l}^{1/l(c_1)}{K_o}^{1/l(c_1)}\Big(\frac{|f(A_0)|}{|Df^{k_1}(f(c_1))|}\Big)^{1/l(c_1)} \quad (one\ side\ Koebe\ principle) \\
&\le 3{K_l}^{2/l(c_1)}{K_o}^{1/l(c_1)}\Big(\frac{|A_0|^{l(c_0)}}{|Df^{k_1}(f(c_1))|}\Big)^{1/l(c_1)} \quad (oder\ of\ critical\ point) \\
&= 3{K_l}^{2/l(c_1)}{K_o}^{1/l(c_1)}|A_0|\Big(\frac{|A_0|^{l(c_0)-l(c_1)}}{|Df^{k_1}(f(c_1))|}\Big)^{1/l(c_1)} \\
&\le 3{K_l}^{2/l(c_1)}{K_o}^{1/l(c_1)}|A_0|\Big(\frac{|f^{k_0}(c_1)-c_0|^{l(c_0)-l(c_1)}}{|Df^{k_1}(f(c_1))|}\Big)^{1/l(c_1)} \qquad (3.\ 11) \\
&:= K|A_0|\Big(\frac{|f^{k_0}(c_1)-c_0|^{l(c_0)-l(c_1)}}{|Df^{k_1}(f(c_1))|}\Big)^{1/l(c_1)} \le |A_0| < \sigma,
\end{aligned}
$$

Similarly we can get for $2 \le i < s$,

$$
\begin{aligned}
&|R_i \cup I_i \cup A_i| \le 3|R_i| \le 3{K_l}^{1/l(c_i)}|f(R_i)|^{1/l(c_i)} \\
&\le 3{K_l}^{2/l(c_i)}{K_o}^{1/l(c_i)}\Big(\frac{|f(A_{i-1})|}{|Df^{k_{i-1}}(f(c_i))|}\Big)^{1/l(c_i)} \\
&\le 3{K_l}^{2/l(c_i)}{K_o}^{1/l(c_i)}\Big(\frac{|A_{i-1}|^{l(c_{i-1})}}{|Df^{k_{i-1}}(f(c_i))|}\Big)^{1/l(c_i)} \\
&\le 3{K_l}^{2/l(c_i)}{K_o}^{1/l(c_i)}|R_{i-1}|\Big(\frac{|R_{i-1}|^{l(c_{i-1})-l(c_i)}}{|Df^{k_{i-1}}(f(c_i))|}\Big)^{1/l(c_i)} \\
&\le 3{K_l}^{2/l(c_i)}{K_o}^{1/l(c_i)}|R_{i-1}|\Big(\frac{|f^{k_{i-1}}(c_i)-c_{i-1}|^{l(c_{i-1})-l(c_i)}}{|Df^{k_{i-1}}(f(c_i))|}\Big)^{1/l(c_i)} \\
&:= K|R_{i-1}|\Big(\frac{|f^{k_{i-1}}(c_i)-c_{i-1}|^{l(c_{i-1})-l(c_i)}}{|Df^{k_{i-1}}(f(c_i))|}\Big)^{1/l(c_i)} \\
&\le |R_{i-1}| \le \sigma,
\end{aligned}
$$

where the last three inequalities are followed from the induction assumption ($k_{i-1} > v(\sigma)$ and $|T_{i-1}| \le \sigma$) and the following relation

$$\frac{1}{3}|f^{k_{i-1}}(c_i) - c_{i-1}| \le |R_{i-1}| \le |f^{k_{i-1}}(c_i) - c_{i-1}|.$$

So we finish the proof of the Lemma. □

For every s-tuple $(k_0, k_1, ..., k_{s-1})$, there are at most $(2\sharp C)^s$ intervals I such that $f^{n_s}(I)$ slides to the same interval G. Furthermore, by Lemma 2, it follows that $k_i \ge v(\sigma)$ for all $0 \le i < s$ from the definition of $v(\sigma)$. Therefore, by (3. 8), (3. 9) and (3. 10), there exists a constant $K_S > 4$

(depending on K_o) such that

$$\sum_{I\in\mathcal{S}_n(c)}|I|\le\sum_{c_s\in C}\sum_{k_j\ge v(\sigma),\sum_j k_j=n-n_s\le n}(2\sharp C)^s|E_{n_s}(c_s,\frac{K_S}{2}|I_s|)|. \tag{3. 12}$$

The transport case: Suppose $I\in\mathcal{T}_n(c)$, by the definition of $\mathcal{T}_n(c)$, f^n has at least one critical point in $I'\supset I$. Let $k<n$ be the maximal integer such that $f^k(I')$ contains a critical point $\tilde{c}$, and denote the set of such kind of intervals by $\mathcal{T}_n^k(\tilde{c},c)$. Clearly, f^{n-k-1} maps $f^{k+1}(I')$ diffeomorphically into $B(c,3\epsilon)$.

Lemma 3. *There exists $K_{\mathcal{T}}>0$ such that*

$$\begin{aligned}\sum_{I\in\mathcal{T}_n(c)}|I| &\le \sum_{n-k\ge v(\sigma)}\sum_{\mathcal{T}_n^k(\tilde{c},c)}|I| \\ &\le \sum_{\tilde{c}\in C}\sum_{n-k\ge v(\sigma)}|E_k\Big(\tilde{c},\quad K_{\mathcal{T}}\epsilon\frac{|f^{n-k}(\tilde{c})-c|^{(l(c)-l(\tilde{c}))/l(\tilde{c})}}{|Df^{n-k}(f(\tilde{c}))|^{1/l(\tilde{c})}}\Big)|.\end{aligned} \tag{3. 13}$$

Proof. Observe that f^{n-k-1} maps $f^{k+1}(I')$ diffeomorphically into $B(c,3\epsilon)$, $f(\tilde{c})\in f^{k+1}(I')\subset[x,y]$, where f^{n-k-1} maps $[x,y]$ diffeomorphically onto $B(c,3\epsilon)$, and $B(c,3\epsilon)$ contains a 1-scaled neighborhood of $B(c,\epsilon)$. One can get from the one-side Koebe principle that there exists K_o such that

$$\frac{|f^n(I)|}{|f^{k+1}(I)|}\ge K_o|Df^{n-k-1}(f(\tilde{c}))|,$$

i.e.,

$$|f^{k+1}(I)|\le K_o\Big(\frac{|f^n(I)|}{|Df^{n-k-1}(f(\tilde{c}))|}\Big).$$

From the orders of the critical points, it follows that

$$|f^k(I)|\le K_l{K_o}^{1/l(\tilde{c})}\Big(\frac{|f^n(I)|}{|Df^{n-k-1}(f(\tilde{c}))|}\Big)^{1/l(\tilde{c})}.$$

Since $f^{n-k}(\tilde{c})\in B(c,3\epsilon)$, the Chain Rules and orders of the critical points indicate that

$$\begin{aligned}|Df^{n-k}(f(\tilde{c}))| &= |Df^{n-k-1}(f(\tilde{c}))||Df(f^{n-k-1}(f(\tilde{c})))| \\ &\le K_l|Df^{n-k-1}(f(\tilde{c}))||f^{n-k-1}(f(\tilde{c}))-c|^{l(c)-1}.\end{aligned}$$

Therefore, there exists a constant K_3(depending K_o and K_l) such that

(3. 14)

$$\begin{aligned}|f^k(I)| &\le K_l{K_o}^{1/l(\tilde{c})}\Big(\frac{K_l|f^n(I)|}{|Df^{n-k}(f(\tilde{c}))||f^{n-k}(\tilde{c})-c|^{1-l(c)}}\Big)^{1/l(\tilde{c})} \\ &= K_l{K_o}^{1/l(\tilde{c})}\Big(\frac{K_l|f^n(I)||f^{n-k}(\tilde{c})-c|^{l(c)-l(\tilde{c})}|f^{n-k}(\tilde{c})-c|^{l(\tilde{c})-1}}{|Df^{n-k}(f(\tilde{c}))|}\Big)^{1/l(\tilde{c})} \\ &\le K_3\epsilon\Big(\frac{|f^{n-k}(\tilde{c})-c|^{l(c)-l(\tilde{c})}}{|Df^{n-k}(f(\tilde{c}))|}\Big)^{1/l(\tilde{c})}.\end{aligned}$$

Here the last inequality follows from the following relationship

$$|f^n(I)| \leq 2\epsilon,\ l(\tilde{c}) \geq 1 \ \ and\ |f^{n-k}(\tilde{c}) - c| \leq 3\epsilon.$$

Since the number of critical points is finite and inequality (3. 14), we know that there exists a constant $K_{\mathcal{T}}$ such that for all critical point $\tilde{c} \in C$,

$$\begin{aligned} I &\subset f^{-k}(f^k(I)) \\ &\subset E_k\Big(\tilde{c},\ \ K_{\mathcal{T}}\epsilon\frac{|f^{n-k}(\tilde{c}) - c|^{l(c)/l(\tilde{c})-1}}{|Df^{n-k}(f(\tilde{c}))|^{1/l(\tilde{c})}}\Big). \end{aligned}$$

Since the definition of $v(\sigma)$ implies $n - k > v(\sigma)$, summing over all such I gives Lemma 2.

□

We proceed to show Proposition 1, by Lemma 4.9 in [12] and the summability conditions, one can choose σ so small that for $n > 1$,

(3. 15)

$$\sum_{c_s\in C}\ \sum_{k_j\geq v(\sigma),\sum_j k_j=n-n_s\leq n} 3K_S\prod_{i=0}^{s-1}(2\sharp C)K_2\Big(\frac{|f^{k_i}(c_{i+1}) - \tilde{c}|^{l(\tilde{c})-l(c_{i+1})}}{|Df^{k_i}(f(c_{i+1}))|}\Big)^{1/l(c_{i+1})} \leq 1,$$

and

$$\sum_{\tilde{c}\in C}\ \sum_{n-k\geq v(\sigma)} 3K_{\mathcal{T}}\Big(\frac{|f^{n-k}(\tilde{c}) - c|^{l(c)-l(\tilde{c})}}{|Df^{n-k}(f(\tilde{c}))|}\Big)^{1/l(\tilde{c})} \leq 1. \tag{3. 16}$$

So, for $n - k \geq v(\sigma)$, we have the following inequalities,

$$\frac{K_S}{2}|I_s| \leq \frac{\sigma}{3}, \tag{3. 17}$$

and

$$K_{\mathcal{T}}\epsilon\Big(\frac{|f^{n-k}(\tilde{c}) - c|^{l(c)-l(\tilde{c})}}{|Df^{n-k}(f(\tilde{c}))|}\Big)^{1/l(\tilde{c})} \leq \frac{\sigma}{3}. \tag{3. 18}$$

With the notations from the beginning of the proof, we have

$$E_n(c,\epsilon) = \bigcup_{I\in\mathcal{R}_n} I \cup \bigcup_{I\in\mathcal{S}_n} I \cup \bigcup_{I\in\mathcal{T}_n} I,$$

and

$$|E_n(c,\epsilon)| \leq \sum_{I\in\mathcal{R}_n}|I| + \sum_{I\in\mathcal{S}_n}|I| + \sum_{I\in\mathcal{T}_n}|I|.$$

Therefore, by (3. 4), (3. 12), (3. 13), (3. 17), and (3. 18), and using the induction hypothesis we have

$$\begin{aligned} &|E_n(c,\epsilon)| \leq \frac{K_{\mathcal{R}}}{\sigma}\epsilon\ \ + \\ &\sum_{c_s\in C}\ \sum_{k_j\geq v(\sigma),\sum_j k_j=n-n_s\leq n} K_S\frac{3K_R}{\sigma}\epsilon\prod_{i=0}^{s-1}K_2(2\sharp C)\Big(\frac{|f^{k_i}(c) - \tilde{c}|^{l(\tilde{c})-l(c)}}{|Df^{k_i}(f(c_i))|}\Big)^{1/l(c)} \\ &\ + \sum_{\tilde{c}\in C}\ \sum_{n-k\geq v(\sigma)} K_{\mathcal{T}}\frac{3K_R}{\sigma}\epsilon\frac{|f^{n-k}(\tilde{c}) - c|^{l(c)/l(\tilde{c})-1}}{|Df^{n-k}(f(\tilde{c}))|^{1/l(\tilde{c})}}. \end{aligned}$$

Then by the choice of σ, and by (3. 15), and (3. 16), we obtain for $\epsilon < \frac{\sigma}{3}$,

$$|E_n(c,\epsilon)| \leq \frac{3K_{\mathcal{R}}}{\sigma}\epsilon,$$

which completes the proof. $\square$

Proposition 2: Let A be any measurable set, then there is a constant $K_5 > 0$ such that

$$|f^{-n}(A)| \leq K_5|A|^{1/l_{\max}}.$$

Proof. Since Proposition 1 says that there is a control of the measure of the preimages of a small neighborhood containing a critical point, the proof can be divided into two parts. The first part is to bound the measure of the preimages of a small intervals "at the end of branches" by the measure of the preimages of a small neighborhood containing a critical point via the following Lemma.

Lemma 4. *(Lemma 1.2.1 in* [7]*) Let $f \in \mathcal{A}$, there exists $K_4 > 0$ such that any interval I for which $f^n|_I$ is monotone and continuous, $m(f^n(I)) \leq \epsilon$ and one of the boundary points of I is a critical point of f^n, we have*

$$I \subset E_i(c, K_4\Big(\frac{\epsilon}{|Df^{n-i-1}(f(c))|}\Big)^{1/l(c)})$$

for some $0 \leq i \leq n (i = n$ put $Df^{n-i-1}(f(c)) = 1$) and $c \in C$.

The second part is to relate the measure of the preimages of any small Borel sets to the measure of the preimages of the intervals "at the end of the branches" satisfying Lemma 3 by using the Minimum principle. We only consider the case $|A| = \epsilon > 0$, let $J = (a, b)$ be a branch of f^n, denote $\mathcal{J}$ by the all of the branches of f^n, choose $J_- = (a, d_1)$, and $J_+ = (d_2, b)$ satisfies $f^n(J_\pm) = \epsilon$. By the Minimum principle (Lemma 4.2 in [12]), it follows that

$$|f^{-n}(A) \cap J| \leq K_o|J_- \cup J_+|.$$

On the other hand, Lemma 4 indicates that

$$J_\pm \subset E_{i^\pm}\Big(c^\pm, K_4\frac{1}{|Df^{n-i^\pm-1}(f(c^\pm))|^{1/l(c^\pm)}}\epsilon^{1/l(c^\pm)}\Big),$$

where $c^+ = f^{i^+}(a)$, and $c^- = f^{i^-}(b)$. Thus, by Proposition 1, it follows that

$$\begin{aligned}
|f^{-n}(A)| &\leq \sum_{J\in\mathcal{J}} |f^{-n}(A) \cap J| \leq \sum_{J\in\mathcal{J}} K_o|J_- \cup J_+| \\
&\leq \sum_{J\in\mathcal{J}} K_o|E_{i^\pm}(c^\pm, K_4\Big(\frac{1}{|Df^{n-i^\pm-1}(f(c^\pm))|}\Big)^{\frac{1}{l(c^\pm)}}\epsilon^{1/l(c^\pm)})| \\
&\leq 2\sum_{c\in C}\sum_{i=0}^{n-1} K_oK_1K_4\Big(\frac{1}{|Df^{n-i-1}(f(c))|}\Big)^{1/l(c)}\epsilon^{1/l(c)} \\
&\leq K_5\epsilon^{1/l_{\max}},
\end{aligned}$$

where $l_{\max}$ is the maximum of the orders of the critical points, the last inequality follows from the second summability condition (1. 2). This implies Proposition 2.

□

4. PROOF OF THEOREMS

The proof of theorem A. We argue by contradiction. Suppose there exists a wandering interval J for $f \in \mathcal{A}$. Then we have $\sum_{n=0}^{\infty} |f^n(J)| < \infty$. Since $J, f(J), \cdots, f^n(J), \cdots$ are disjoint, $|f^n(J)| \to 0$ as $n \to +\infty$. By Proposition 2, we get

$$|J| \le |f^{-n}(f^n(J))| \le K_5 |f^n(J)|^{1/l_{\max}}, \quad \forall\ n > 0.$$

Therefore, let $n \to \infty$, the above inequality implies that $|J| = 0$, which contradicts to J is a wandering interval. Theorem A is proved. □

The proof of theorem B. Let $f \in \mathcal{A}$, denote P_f as the Fronbenius-Perron operator induced by f. We consider the sequence of functions $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$, where $\mathbf{1}$ is the constant function on the unit interval M. By Proposition 2, it follows that $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$ is uniformly integrable, i.e., for every $\epsilon > 0$, there is $\delta > 0$ such that

$$\int_A P_f^n \mathbf{1} dx = \int_{f^{-n}(A)} \mathbf{1} dx \le K_5 |A|^{1/l_{\max}} \le \epsilon,$$

if $m(A) \le \delta$, $n > 0$. On the other hand, since the L^1 norm of $P_f^n \mathbf{1}$ is equal to 1, by a result from functional analysis [14], $\{P_f^n \mathbf{1}\}_{n=0}^{\infty}$ is weakly precompact in L^1. The same applies to the sequence $\{g_n := \frac{1}{n}\sum_{j=0}^{n-1} P_f^j \mathbf{1}\}$, there is a subsequence g_{n_k} that converges weakly to g_*, $P_f g_* = g_*$. By the abstract ergodic theorem of Kakutani and Yosida, g_n converges strongly to g_*, and g_* is an invariant density of f [19]. As a result, the measure defined by

$$\mu(A) = \int_A g_*(x) dx, \quad A \textit{ is a Borel set}$$

is an invariant probability measure of f.

By Proposition 2, we have $\mu(A) \le K_5 m(A)^{1/l_{\max}}$. If $l_{\max} > 1$, then the probability density g_* is a L^p function for $1 \le p < l_{\max}/(l_{\max} - 1)$. □

The proof of theorem C. For a small δ, denote by $N_\delta(c) := \{x; |f(x) - f(c)| < \delta\}$ a small neighborhood containing c, and $N_\delta := \cup_{c \in C} N_\delta(c)$. Assume that $x, f(x), ..., f^{n-1}(x) \notin N_\delta$, and $f^n(x) \in N_\delta$, let $T_n(x)$ be the maximal interval containing x on which f^n is a diffeomorphism.

Claim: If $f^n(x) \in N_\delta(c)$, then $B(f^n(x), |N_\delta(c)|) \subset f^n(T_n(x))$.

Indeed, let $T_n(x) = (a, b)$, assume without loss of generality that $|f^n(x) - f^n(a)| \le |f^n(b) - f^n(x)|$, we then assume $|f^n(x) - f^n(a)| \le |N_\delta(c)|$ by contradiction. So there is a critical point $c_j \in C$ and $0 \le n_1 < n$ such that $f^{n_1}(a) = c_j$. The orders of the critical point imply

$$|f^{n-n_1}(c_j) - c| \le 2K_l \delta^{1/l(c)}. \tag{4. 19}$$

Observe that $x, f(x), ..., f^{n-1}(x) \notin N_\delta$ and $f^{n_1}(x)$ lies in a small neighborhood of c_j, then we have $|f^{n_1+1}(x) - f(c_j)| \ge \delta$. Because f^{n-n_1-1} is

a diffeomorphism on $(f^{n_1+1}(x), f(c_j))$, by the mean value theorem, there exists $z \in (f^{n_1+1}(x), f(c_j))$ such that

$$|Df^{n-n_1-1}(z)| = \frac{|f^{n-n_1}(c_j) - f^n(x)|}{|f^{n_1+1}(x) - f(c_j)|} \leq \frac{K_l \delta^{1/l(c)}}{\delta} = K_l \delta^{-1+1/l(c)}.$$

On the other hand, since $|f^n(x) - f^n(a)| \leq |f^n(b) - f^n(x)|$ and $f^{n-n_1-1}(z) \in (f^n(a), f^n(x))$, the one-side Koebe principle indicates

$$|Df^{n-n_1-1}(z)| \geq K_o |Df^{n-n_1-1}(f(c_j))|.$$

Therefore, we obtain that

$$\begin{aligned} |Df^{n-n_1}(f(c_j))| &= |Df^{n-n_1-1}(f(c_j))||Df(f^{n-n_1}(c_j))| \\ &\leq \frac{K_l}{K_o} |Df^{n-n_1-1}(z)||f^{n-n_1}(c_j) - c|^{l(c)-1} \\ &\leq \frac{K_l}{K_o} K_l \delta^{-1+1/l(c)} \big(2K_l \delta^{1/l(c)}\big)^{l(c)-1} = \frac{K_l}{K_o} 2^{l(c)-1} {K_l}^{l(c)}. \end{aligned}$$

Since $|Df^n(f(c_j))| \to \infty$ as $n \to \infty$, the above estimation implies that $n - n_1$ is bounded above by some integer n_0 which does not depend on δ. On the other hand, the (4. 19) indicates that these exists $\delta_0 > 0$ such that for any $\delta < \delta_0$, one gets $n - n_1 > n_0$. So we obtain a contradiction. The Claim is true.

Next we assume $y \in T_n(x)$ such that $f^n(y) = c$, then there exists $K_1 > 0$ such that $|Df^n(y)| > K_1$ by Proposition 1. On the other hand, $f^n(x) \in N_\delta(c)$ and the above claim give

$$|f^n(x) - c| \leq |N_\delta(c)| \quad \textit{and} \quad d(f^n(x), f^n(\partial T_n(x))) \geq |N_\delta(c)|.$$

Therefore, by the one-side Koebe principle, it follows that there exists $K_6 > 0$ (K_6 is not depending on δ) such that

$$|Df^n(x)| \geq K_o |Df^n(y)| \geq K_o K_1 \geq K_6.$$

□

Remark 3. We only need the $|Df^n(f(c_j))| \to \infty$ as $n \to \infty$ and result in Proposition 1 in the proof of Theorem C, a stronger version of BBC property of multimodal maps with equal critical orders from two sides under a stronger increasing condition can be found in [10].

References

[1] V. Araújo, S. Luzzatto, and M. Viana. Invariant measures for interval maps with critical points and singularities. *Adv. Math.*, 221(5):1428–1444, 2009.

[2] D. Berry and B. D. Mestel. Wandering intervals for Lorenz maps with bounded nonlinearity. *Bull. London Math. Soc.*, 23(2):183–189, 1991.

[3] A. M. Blokh and M. Yu. Lyubich. Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case. *Ergodic Theory Dynam. Systems*, 9(4):751–758, 1989.

[4] A. M. Blokh and M. Misiurewicz. Collet-Eckmann maps are unstable. *Comm. Math. Phys.*, 191(1):61–70, 1998.

[5] H. Bruin, S. Luzzatto, and S. Van Strien. Decay of correlations in one-dimensional dynamics. *Ann. Sci. École Norm. Sup. (4)*, 36(4):621–646, 2003.

[6] H. Bruin, J. Rivera-Letelier, W. Shen, and S. van Strien. Large derivatives, backward contraction and invariant densities for interval maps. *Invent. Math.*, 172(3):509–533, 2008.

[7] H. Bruin and S. van Strien. Existence of absolutely continuous invariant probability measures for multimodal maps. In *Global analysis of dynamical systems*, pages 433–447. Inst. Phys., Bristol, 2001.
[8] H. Bruin and S. van Strien. Expansion of derivatives in one-dimensional dynamics. *Israel J. Math.*, 137:223–263, 2003.
[9] R. Camelier and C. Gutierrez. Affine interval exchange transformations with wandering intervals. *Ergodic Theory Dynam. Systems*, 17(6):1315–1338, 1997.
[10] S. Cedervall. Invariant measures and correlation decay for s-multimodal interval maps. *Ph.D.thesis*, page 111, 2006.
[11] W. de Melo and S. van Strien. A structure theorem in one-dimensional dynamics. *Ann. of Math. (2)*, 129(3):519–546, 1989.
[12] W. de Melo and S. van Strien. *One-dimensional dynamics*, volume 25 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1993.
[13] K. Díaz-Ordaz, M. P. Holland, and S. Luzzatto. Statistical properties of one-dimensional maps with critical points and singularities. *Stoch. Dyn.*, 6(4):423–458, 2006.
[14] N. Dunford and J.T. Schwartz. *Linear Operators. Part I: General Theory.* Wily, New York, 1957.
[15] J. Guckenheimer. Sensitive dependence to initial conditions for one-dimensional maps. *Comm. Math. Phys.*, 70(2):133–160, 1979.
[16] C. Gutierrez, S. Lloyd, and B. Pires. Affine interval exchange transformations with flips and wandering intervals. *Proc. Amer. Math. Soc.*, 137(4):1439–1445, 2009.
[17] G. Hall. A C^∞ Denjoy counterexample. *Ergodic Theory Dynamical Systems*, 1(3):261–272 (1982), 1981.
[18] O. S. Kozlovski. Getting rid of the negative Schwarzian derivative condition. *Ann. of Math. (2)*, 152(3):743–762, 2000.
[19] A. Lasota and M. C. Mackey. *Chaos, fractals, and noise*, volume 97 of *Applied Mathematical Sciences*. Springer-Verlag, New York, second edition, 1994. Stochastic aspects of dynamics.
[20] M. Yu. Lyubich. Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. I. The case of negative Schwarzian derivative. *Ergodic Theory Dynam. Systems*, 9(4):737–749, 1989.
[21] M. Martens and W. de Melo. Universal models for Lorenz maps. *Ergodic Theory Dynam. Systems*, 21(3):833–860, 2001.
[22] M. Martens, W. de Melo, and S. van Strien. Julia-Fatou-Sullivan theory for real one-dimensional dynamics. *Acta Math.*, 168(3-4):273–318, 1992.
[23] R. J. Metzger. Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 17(2):247–276, 2000.
[24] M. Misiurewicz. Absolutely continuous measures for certain maps of an interval. *Inst. Hautes Études Sci. Publ. Math.*, (53):17–51, 1981.
[25] T. Nowicki and S. van Strien. Invariant measures exist under a summability condition for unimodal maps. *Invent. Math.*, 105(1):123–136, 1991.
[26] A. Rovella. The dynamics of perturbations of the contracting Lorenz attractor. *Bol. Soc. Brasil. Mat. (N.S.)*, 24(2):233–259, 1993.
[27] A. J. Schwartz. A generalization of a Poincaré-Bendixson theorem to closed two-dimensional manifolds. *Amer. J. Math. 85 (1963), 453-458; errata, ibid*, 85:753, 1963.
[28] M. St. Pierre. Topological and measurable dynamics of Lorenz maps. *Dissertationes Math. (Rozprawy Mat.)*, 382:134, 1999.
[29] D. Sullivan. Quasiconformal homeomorphisms and dynamics. I. Solution of the Fatou-Julia problem on wandering domains. *Ann. of Math. (2)*, 122(3):401–418, 1985.
[30] S. van Strien and E. Vargas. Real bounds, ergodicity and negative Schwarzian for multimodal maps. *J. Amer. Math. Soc.*, 17(4):749–782 (electronic), 2004.
[31] J.-C. Yoccoz. Il n'y a pas de contre-exemple de Denjoy analytique. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(7):141–144, 1984.